Eur. Phys. J. C 42, 243–251 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02276-y

THE EUROPEAN
PHYSICAL JOURNAL C

Green's function for spinless particle via Parisi–Wu stochastic quantization method

Z. Lehtihet, L. Chetouani[a]

Département de physique, Faculté des Sciences, Université Mentouri, 25000 Constantine, Algeria

Received: 27 January 2005 / Revised version: 3 April 2005 /
Published online: 8 June 2005 –

Abstract. An exact and analytic Green function for a spinless particle in interaction with an electromagnetic plane wave field, expressed in the coordinate gauge is given by Parisi–Wu stochastic quantization method. We separate the classical calculations from those related to the quantum fluctuation term. We have used a perturbative treatment relying on phase and configuration spaces formulation.

PACS. 03.65.Ca, 03.65.Pm, 05.10.Gg

1 Introduction

At the present time, there is another approach equivalent to the standard quantum mechanics which is the stochastic quantization method (SQM) introduced by Parisi and Wu [1]. This approach enables us to deal with a larger class of dynamical systems and holds the attention of the physicists thanks to its practical and technical merits in field theory. The idea is to combine quantum field theory and statistical mechanics. Then the path integral for SQM has been formulated relying on the analogy between Feynman's measure and the thermal equilibrium distribution [2]. Briefly, the formulation is mainly based on the introduction of a fictitious time u in addition to the real one and of a noise which is generally selected white. All the results of the standard quantum mechanics agree with those obtained by the SQM ($u \to \infty$) [3]. Thus we can represent Green's functions as the correlation functions of a statistical system in thermal equilibrium. Although this method appears to be powerful, its application in non-relativistic quantum mechanics remains rare. To our knowledge, there is a reduced list of exact calculations of the transition amplitude which are related to the quadratic action. Thus, the propagator was given in the phase and configuration spaces for the case of the non-relativistic free particle, the harmonic force, the constant magnetic field and the free Grassman case [4]. To incorporate the general form of interaction, Nakazato [5] supplemented it by a perturbative treatment.

Our purpose in this paper is to apply this perturbative treatment of SQM to the case of a relativistic particle without spin, in interaction with a plane wave field in coordinate gauge [6]. This problem has already been investigated via the Schwinger formalism [7] by solving the equations of motion in the Heisenberg picture; second quantization by using the Furry transformations [8] and finally, via the path integral framework [9] where it was shown that the semi-classical calculation is still exact. Here the same problem is reconsidered by using the stochastic quantization method. The Green's function calculation will be very simple.

Let us expose the configuration of the plane wave field. Thus, the quadripotential A_μ is chosen in the coordinate gauge:

$$(x-x_0)^\mu A_\mu(x) = 0, \tag{1}$$

where x_0 is an arbitrary reference point.

The advantage of this gauge choice is that having the electromagnetic tensor $F_{\mu\nu}$, the 4-vector potential A_μ is determined in a unique way following the inversion formula [10]

$$A_\mu(x) = \int_0^1 \mathrm{d}\alpha\alpha\,(x-x_0)^\nu F_{\mu\nu}(\alpha x)\,, \tag{2}$$

with the electromagnetic plane wave tensor having the form

$$F_{\mu\nu}(x) = f_{\mu\nu} F(\xi)\,, \tag{3}$$

where $\xi = \eta x$, $F(\xi)$ is an arbitrary function of ξ and $f_{\mu\nu}$ is a constant antisymmetric tensor verifying with η_μ the following useful properties:

$$\begin{gathered} \eta^\mu \eta_\mu = 0, \qquad \eta_\mu f^{\mu\nu} = 0, \qquad \eta_\mu f^{\mu\nu *} = 0, \\ f^*_{\mu\lambda} f^\lambda_\nu = 0, \qquad f^*_{\mu\lambda} f^{\lambda *}_\nu = f_{\mu\lambda} f^\lambda_\nu = \eta_\mu \eta_\nu, \end{gathered} \tag{4}$$

where f^* is the dual tensor of f.

In this gauge, the 4-potential A_μ will evidently take the following form:

$$A_\mu(x) = f_{\mu\nu}(x-x_0)^\nu K(\xi,\xi_0)\,, \tag{5}$$

[a] e-mail: chetoua1@caramail.com

where $K(\xi, \xi_0)$ satisfies the equation

$$2K + (\xi - \xi_0)\frac{\mathrm{d}K}{\mathrm{d}\xi} = -F(\xi), \tag{6}$$

which has the following solution:

$$K(\xi, \xi_0) = -\frac{A(\xi)}{\xi - \xi_0} + \frac{1}{(\xi - \xi_0)^2}\int_{\xi_0}^{\xi} \mathrm{d}\eta A(\eta), \tag{7}$$

with $\mathrm{d}A/\mathrm{d}\xi = F(\xi)$, x being a point of the four-dimensional space-time endowed with the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and p^μ the four-dimensional momentum vector $p^\mu = (p^0, p^1, p^2, p^3)$. To simplify the calculations we choose the reference point x_0 equal to $x_i = x(s_i)$.

Let us give the Klein–Gordon equation (KG) for a spinless particle interacting with a plane wave field:

$$\left[(\mathrm{i}\partial_\mu - eA_\mu)(\mathrm{i}\partial^\mu - eA^\mu) - m^2\right]\Delta(x_f, x_i) = \delta^4(x_f - x_i), \tag{8}$$

where $\Delta(x_f, x_i)$ is the Green's function and λ is the proper time [11,7] given by

$$\lambda = s_f - s_i; \tag{9}$$

in this case, the Green's function $\Delta(x_f, x_i)$ becomes

$$\Delta(x_f, x_i) = \frac{1}{2\mathrm{i}}\int_0^\infty \mathrm{d}\lambda \exp\left[-\frac{\mathrm{i}m^2\lambda}{2}\right]\overline{K}(x_f, x_i; \lambda), \tag{10}$$

where $\overline{K}(x_f, x_i; \lambda)$ is the propagator which satisfies the following equation:

$$\left[\mathrm{i}\frac{\partial}{\partial\lambda} - \widehat{H}\right]\overline{K}(x_f, x_i; \lambda) = \delta(\lambda)\,\delta^4(x_f - x_i), \tag{11}$$

in our case, the hamiltonian operator $\widehat{H}$ is given by $\widehat{H} = -\frac{1}{2}(\widehat{p} - eA)^2$ with $p_\mu = \mathrm{i}\partial_\mu$.

In this paper, we show that the exact Green's function for a Klein–Gordon particle interacting with a plane wave can be obtained in coordinate gauge relying on the stochastic quantization method of Parisi and Wu [4,5]. In Sect. 2, we review the formulation of the SQM. In Sect. 3, we treat the problem of the plane wave relying on the phase space in the framework of SQM, we calculate perturbatively the classical action and also find the fluctuation factor via the Langevin equations iteratively. In Sect. 4, we rely on the configuration space to calculate once more, perturbatively, the classical action and the fluctuation factor, the latter is obtained when we take the equilibrium limit. Finally, in Sect. 5, the propagator is found relying on the configuration space and the exact Green's function is determined.

2 Review of the stochastic quantization method

Let us give basic principles of the SQM of Parisi–Wu [1]. We obtain the quantum mechanics results as the thermal equilibrium limit of a hypothetical stochastic process. The dynamical variable $x(s)$ is assumed to be a stochastic one $x(s, u)$, and we introduce a new fictitious time u via the Langevin equation given by

$$\frac{\partial}{\partial u}x(s, u) = \mathrm{i}\frac{\delta S}{\delta x(s, u)} + \varrho(s, u), \tag{12}$$

where S is the classical action of the system and ϱ is the Gaussian white noise, characterized by

$$\langle\varrho(s, u)\rangle = 0,$$
$$\langle\varrho(s, u)\varrho(s', u')\rangle = 2\delta(s - s')\delta(u - u'). \tag{13}$$

The Langevin equation (12) is solved under boundary conditions to get $x(s, u)$ as a functional of the noise. We calculate the equal time correlation function $\langle x(s_1, u)\, x(s_2, u) ...\rangle$ by using (13) and taking the equilibrium limit ($u \to \infty$.).

We can also express in the Fokker–Planck picture the stochastic average $\langle x(s_1, u)\, x(s_2, u) ...\rangle$ which is given by the functional integral

$$\langle x(s_1, u)\, x(s_2, u) ...\rangle = \int Dx x(s_1)\, x(s_2) ... P[x; u], \tag{14}$$

where $P[x, u]$ is the probability distribution which satisfies the Fokker–Planck equation

$$\frac{\partial}{\partial u}P[x, u] = \int_{-\infty}^{+\infty} \mathrm{d}s \frac{\delta}{\delta x(s)}\left(\frac{\delta}{\delta x(s)} - \mathrm{i}\frac{\delta S}{\delta x(s)}\right)P[x, u], \tag{15}$$

and is normalized as

$$\langle 1\rangle = \int DxP[x, u] = 1. \tag{16}$$

Therefore, the stationary solution P is given by $\exp\left(\frac{\mathrm{i}S}{\hbar}\right)$ as $u \to \infty$ and the correlation function becomes exactly the same as that defined by the Feynman path integral formalism:

$$\lim_{u\to\infty}\langle x(s_1, u)\, x(s_2, u) ...\rangle = \frac{\int Dx x(s_1)\, x(s_2) ... \mathrm{e}^{\mathrm{i}S}}{\int Dx \mathrm{e}^{\mathrm{i}S}} = \langle 0 \mid Tx(s_1)\, x(s_2) ... \mid 0\rangle. \tag{17}$$

Now, let us deduce the transition amplitude [4]. We evaluate the correlation function under the boundary conditions

$$x(s_i, u) = x_i, \qquad x(s_f, u) = x_f, \tag{18}$$

using (18) and the normalization condition (16); as we take the limit $u \to \infty$, the stochastic average becomes

$$\lim_{u\to\infty}\langle x(s_1, u)\, x(s_2, u) ...\rangle = \frac{\int_{x(s_1)=x_i}^{x(s_f)=x_f} Dx x(s_1)\, x(s_2) ... \mathrm{e}^{\mathrm{i}S}}{\int_{x(s_1)=x_i}^{x(s_f)=x_f} Dx \mathrm{e}^{\mathrm{i}S}}$$

$$= \frac{\langle 0 \mid Tx(s_1)\, x(s_2) \ldots \mid 0\rangle}{\langle x_f, s_f \mid x_i, s_i\rangle}. \tag{19}$$

We define the state vector by

$$|x_i, s_i\rangle = T\exp\left[\mathrm{i}\int^{s_i} \widehat{H}(s)\mathrm{d}s\right]|x_i, s_i\rangle\,, \tag{20}$$

which is the solution of the following evolution equation:

$$\frac{\partial}{\partial s_i}|x_i, s_i\rangle = \mathrm{i}\widehat{H}(s_i)\,|x_i, s_i\rangle\,, \tag{21}$$

where $\widehat{H} = -\frac{1}{2}\left(\widehat{p} - eA\right)^2$.

The transition amplitude $\langle x_f, s_f \mid x_i, s_i\rangle$ can be related to the Hamiltonian average [4]:

$$\begin{aligned}\overline{K}(x_f, x_i; \lambda) &= \langle x_f, s_f \mid x_i, s_i\rangle \qquad (22)\\ &= c\exp\left[\mathrm{i}\int^{s_i} \lim_{u\longrightarrow\infty}\langle H(s_i, u)\rangle\,\mathrm{d}s_i\right];\end{aligned}$$

in our case $H = -\frac{1}{2}\left(p - eA\right)^2$ and the constant c is s_i independent. It can be fixed by taking a limit as $s_i = s_f$ and imposing the condition

$$\lim_{\lambda\longrightarrow 0}\langle x_f, s_f \mid x_i, s_i\rangle = \delta(x_f - x_i), \quad \lambda = (s_f - s_i)\,. \tag{23}$$

Let us proceed in the same way as Hüffel [4]. We carry out the following decomposition by separating the classical and quantum variables in the expression of the Hamiltonian average. Hence, we obtain two parts, a classical one independent of the fictitious time u and the other one dependent on the noise ϱ and the time u

$$\langle H(s_i, u)\rangle = \langle H_{\mathrm{cl}}(s_i)\rangle + \langle H_Q(s_i, u)\rangle\,. \tag{24}$$

In our case, we define the Hamiltonian for a spinless particle in interaction with an electromagnetic plane wave field by

$$H(s_i, u) = -\frac{1}{2}\left[p(s_i, u) - eA(x(s_i, u))\right]^2; \tag{25}$$

therefore, the transition amplitude takes the following form:

$$\begin{aligned}&\langle x_f, s_f \mid x_i, s_i\rangle \qquad (26)\\ &= c\exp\left[\mathrm{i}S_{\mathrm{cl}}\right]\exp\left[\mathrm{i}\int^{s_i}\lim_{u\longrightarrow\infty}\langle H_Q(s_i, u)\rangle\,\mathrm{d}s_i\right],\end{aligned}$$

where the first term, $\exp\left[\mathrm{i}S_{\mathrm{cl}}\right]$, related to the classical path, is a consequence of the relation [12]

$$\frac{\partial S_{\mathrm{cl}}}{\partial s_i} = H_{\mathrm{cl}}(s_i), \tag{27}$$

and the second exponential factor term contains all the quantum contributions dependent on the noise and the fictitious time u. In fact, this result will enormously simplify the transition amplitude calculations.

Finally, owing to the perturbative treatment, we will calculate the classical action. We use the Langevin equation and work also iteratively to obtain the average $\langle H_Q(s_i, u)\rangle$.

3 Calculation in the phase space formulation

In order to evaluate the transition amplitude, we rely on the space phase formulation of the SQM. First, we set two variables (x_Q^μ, p_Q^μ) where x_Q^μ indicates the deviation of x^μ compared to x_{cl}^μ, the classical path, and p_Q^μ the deviation compared to p_{cl}^μ, the classical momentum,

$$\begin{cases} x^\mu = x_{\mathrm{cl}}^\mu + x_Q^\mu, \\ p^\mu = p_{\mathrm{cl}}^\mu + p_Q^\mu; \end{cases} \tag{28}$$

next, we decompose the plane wave action as

$$S = S_{\mathrm{cl}} + S_Q, \tag{29}$$

where S_{cl} is the classical action given by

$$S_{\mathrm{cl}} = -\int_{s_i}^{s_f}\mathrm{d}s\left[p_{\mathrm{cl}}(s)\frac{\mathrm{d}x_{\mathrm{cl}}(s)}{\mathrm{d}s} - \frac{1}{2}\left[p_{\mathrm{cl}}(s) - eA(\xi)\right]^2\right], \tag{30}$$

and S_Q is the quantum action which includes all the remaining terms as in the following expression:

$$\begin{aligned}S_Q = -\int_{s_i}^{s_f}\mathrm{d}s\Big[&p_{\mathrm{cl}}\frac{\mathrm{d}x_Q}{\mathrm{d}s} + p_Q\frac{\mathrm{d}x_Q}{\mathrm{d}s}\\ &- e\left(A_{\mathrm{cl}} - A(\xi)\right)\left(p_{\mathrm{cl}} + p_Q\right)\\ &- \frac{e^2}{2}\left(A^2(\xi) - A_{\mathrm{cl}}^2\right) - \frac{1}{2}p_Q^2\Big],\end{aligned} \tag{31}$$

with the following boundary conditions:

$$x_Q(s_i) = x_Q(s_f) = 0. \tag{32}$$

We split the Hamiltonian

$$H = H_{\mathrm{cl}} + H_Q, \tag{33}$$

and obtain one classical term

$$H_{\mathrm{cl}} = -\frac{1}{2}\left(p_{\mathrm{cl}} - eA_{\mathrm{cl}}\right)^2, \tag{34}$$

and another one, deduced from (33) and (34), which gives us the remaining terms as the following expression:

$$\begin{aligned}H_Q = &-\frac{1}{2}p_Q^2 - p_{\mathrm{cl}}p_Q - \frac{e^2}{2}A^2(\xi) + e\left(p_{\mathrm{cl}} + p_Q\right)A(\xi)\\ &+ \frac{e^2}{2}A_{\mathrm{cl}}^2 - ep_{\mathrm{cl}}A_{\mathrm{cl}}.\end{aligned} \tag{35}$$

We apply the SQM by introducing a new fictitious time u in the quantum contributions

$$x_Q(s) \longrightarrow x_Q(s, u), \tag{36}$$

$$p_Q(s) \longrightarrow p_Q(s, u); \tag{37}$$

now, the boundary conditions become

$$x_Q(s_i, u) = x_Q(s_f, u) = 0. \tag{38}$$

Let us calculate the classical action.

3.1 Classical action calculations

Before performing the calculations in the phase space, we give the expression of the classical action for a spinless particle in interaction with an electromagnetic plane wave:

$$S_{\rm cl} = -\int_{s_i}^{s_f} {\rm d}s \left[p_{\rm cl}(s) \frac{{\rm d}x_{\rm cl}(s)}{{\rm d}s} - \frac{1}{2}\left(p_{\rm cl}(s) - eA_{\rm cl}(\xi)\right)^2 \right], \tag{39}$$

where the classical Hamiltonian is given by

$$H_{\rm cl} = -\frac{1}{2}\left(p_{\rm cl} - eA_{\rm cl}(\xi)\right)^2 . \tag{40}$$

Now, let us determine the classical path using the Hamilton equations

$$\begin{aligned} \dot p_{{\rm cl}\rho} &= \frac{\partial H_{\rm cl}}{\partial x^\rho} = e\left(p_{{\rm cl}\rho} - eA_{{\rm cl}\rho}\right)\frac{{\rm d}A_{\rm cl}}{{\rm d}x^\rho}, \\ \dot x_{{\rm cl}\rho} &= -\frac{\partial H_{\rm cl}}{\partial p^\rho} = p_{{\rm cl}\rho} - eA_{{\rm cl}\rho}. \end{aligned} \tag{41}$$

We notice that, using (41), it is easy to obtain the classical equation of motion

$$\ddot x = eF\dot x, \tag{42}$$

with the boundary conditions $x_{\rm cl}(s_f) = x_f$, $x_{\rm cl}(s_i) = x_i$.

In order to have the classical path, we perform the integration over the time s. The solution is given by iteration and takes the following form:

$$\begin{aligned} x(s) &= \dot x(s_i)\, s \\ &+ ef\dot x(s_i)\left(\frac{\lambda}{\xi_f - \xi_i}\right)^2 \int_{\xi_i}^{\xi} {\rm d}\zeta' \left[A(\xi') - A(\xi_i)\right] \\ &+ \frac{e^2\eta}{2}\left(\frac{\lambda}{\xi_f - \xi_i}\right)^2 \left(\int_{\xi_i}^{\xi} {\rm d}\xi' \left[A(\xi') - A(\xi_i)\right]^2\right) \\ &+ x(s_i), \end{aligned} \tag{43}$$

where $\zeta = \eta x$ and ${\rm d}A/{\rm d}\xi = F(\xi)$. We have used $\eta f = 0$, $\eta^2 = 0$.

In order to obtain the classical path, we have to calculate the related classical action

$$S = -\int_{s_i}^{s_f} {\rm d}s \left[\frac{1}{2}\left(\frac{{\rm d}x}{{\rm d}s}\right)^2 + eA\left(\frac{{\rm d}x}{{\rm d}s}\right)\right]. \tag{44}$$

From (42), we can see that the square of the velocity and its component on the direction η are preserved during the motion:

$$\dot x^2(s) = \dot x^2(s_i), \quad \eta\dot x(s) = \eta\dot x(s_i). \tag{45}$$

Using $\eta f = 0$, $\eta^2 = 0$ and the properties of the gauge $A(x - x_i) = 0$, $\eta A = 0$, we obtain the following expressions:

$$\begin{aligned} \dot x^2(s) &= \dot x^2(s_i) \\ &= \left(\frac{x_f - x_i}{\lambda}\right)^2 + \frac{e^2}{(\xi_f - \xi_i)^2}\left[\int_{\xi_i}^{\xi_f} {\rm d}\xi A(\xi)\right]^2 \\ &- \frac{e^2}{(\xi_f - \xi_i)}\left(\int_{\xi_i}^{\xi_f} {\rm d}\xi \left[A(\xi)\right]^2\right), \end{aligned} \tag{46}$$

$$\begin{aligned} A(x(s))\,\dot x(s) &= +eK(\xi,\xi_i)\left[\int_{\xi_i}^{\xi} {\rm d}\zeta' A(\xi')\right] \\ &- e(\xi - \xi_i)K(\xi,\xi_i)A(\xi), \end{aligned} \tag{47}$$

which we will insert in (44); then, we perform the integration over s, so we obtain the classical action

$$\begin{aligned} S_{\rm cl} = &-\frac{(x_f - x_i)^2}{2\lambda} + \frac{e^2\lambda}{2(\xi_f - \xi_i)^2}\left[\int_{\xi_i}^{\xi_f} {\rm d}\xi A(\xi)\right]^2 \\ &- \frac{e^2\lambda}{2(\xi_f - \xi_i)}\int_{\xi_i}^{\xi_f} {\rm d}\xi \left[A(\xi)\right]^2 . \end{aligned} \tag{48}$$

Let us now calculate the second term which gives us the fluctuation factor.

3.2 Calculation of $\langle H_Q(s_i, u)\rangle$

We have to calculate the fluctuation factor related to the transition amplitude (26). We perform the calculations in the phase space formulation relying on the stochastic quantization method. In this case, the stochastic variables x_Q and p_Q satisfy the following Langevin equations:

$$\begin{cases} \dfrac{\partial x_Q}{\partial u} = {\rm i}\left[\dfrac{\partial p_Q}{\partial s} + \dfrac{\partial p_{\rm cl}}{\partial s} + \dfrac{1}{2}e^2\eta\dfrac{{\rm d}A^2}{{\rm d}\xi} \right. \\ \qquad \left. - e\eta\dfrac{{\rm d}A}{{\rm d}\xi}\left(p_{\rm cl} + p_Q\right)\right] + \varrho(s,u), \\ \dfrac{\partial p_Q}{\partial u} = {\rm i}\left[p_Q - \dfrac{\partial x_Q}{\partial s} + e\left(A_{\rm cl} - A(\xi)\right)\right] + \chi(s,u), \end{cases} \tag{49}$$

where the white noises fulfil

$$\begin{aligned} \langle \varrho^\mu(s,u)\rangle &= 0, \qquad \langle \chi^\mu(s,u)\rangle = 0, \\ \langle \varrho^\mu(s,u)\varrho_\nu(s',u')\rangle &= \langle \chi^\mu(s,u)\chi_\nu(s',u')\rangle \\ &= 2g^\mu_\nu\delta(s - s')\delta(u - u'). \end{aligned} \tag{50}$$

The Langevin system (49) amounts to the matrix form

$$\begin{aligned} \frac{\partial}{\partial u}\begin{pmatrix} x_Q \\ p_Q \end{pmatrix} &= {\rm i}\begin{pmatrix} 0 & \frac{\partial}{\partial s} \\ -\frac{\partial}{\partial s} & 1 \end{pmatrix}\begin{pmatrix} x_Q \\ p_Q \end{pmatrix} \\ &+ \begin{pmatrix} \dot p_{\rm cl} + \frac{1}{2}e^2\eta\frac{{\rm d}A^2}{{\rm d}\xi} - e\eta\frac{{\rm d}A}{{\rm d}\xi}\cdot\left(p_{\rm cl} + p_Q\right) \\ e\left(A_{\rm cl} - A(\xi)\right) \end{pmatrix} \\ &+ \begin{pmatrix} \varrho(s,u) \\ \chi(s,u) \end{pmatrix}, \end{aligned} \tag{51}$$

which takes the following form:

$$\frac{\partial}{\partial u}\overrightarrow{X} = M\overrightarrow{X} + \overrightarrow{w} + \overrightarrow{v}; \tag{52}$$

we indicate by the vector $\overrightarrow{X}_Q^{(0)}$ [5] the solution of the free equation (without field)

$$\frac{\partial}{\partial u}\overrightarrow{X}_Q^{(0)}(s,u) = M^{(0)}\overrightarrow{X}_Q^{(0)} + \overrightarrow{v}. \tag{53}$$

Formally, the solution of (53) is

$$\overrightarrow{X}_Q^{(0)}(s,u) = \int_{s_i}^{s_f} \mathrm{d}s' \int_{-\infty}^{+\infty} \mathrm{d}u' G(s,u \mid s',u')\, \overrightarrow{v}(s',u'), \tag{54}$$

where $G(s,u \mid s',u')$ is the Green's function associated to the free Langevin system ($A = 0$) and given by

$$G(s,u \mid s',u') = \begin{pmatrix} G_{11}(s,u \mid s',u') & G_{12}(s,u \mid s',u') \\ G_{21}(s,u \mid s',u') & G_{22}(s,u \mid s',u') \end{pmatrix}, \tag{55}$$

which satisfy the free Langevin equation

$$\left(\frac{\partial}{\partial u} - M\right) G(s,u \mid s',u') = \delta(s-s')\,\delta(u-u'), \tag{56}$$

with the boundary conditions

$$G_{11}(s,u \mid s',u') = G_{12}(s,u \mid s',u') = 0, \quad \text{for } s \text{ or } s' = s_i, s_f \text{ or } u < u', \tag{57}$$

and after a long calculation, we obtain the following Green matrix elements:

$$\begin{aligned} G_{11} = &-\theta(u-u')\frac{2i}{\lambda}\sum_{n=1}^{\infty} \sin\frac{n\pi}{\lambda}(s-s_i)\sin\frac{n\pi}{\lambda}(s'-s_i) \\ &\times\left\{\frac{1}{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}\sin\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\right. \\ &\left.+i\cos\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\right\}\exp\frac{i}{2}(u-u'), \end{aligned} \tag{58}$$

$$\begin{aligned} G_{12} = &-\theta(u-u')\frac{4i}{\lambda}\sum_{n=1}^{\infty}\frac{n\pi}{\lambda}\cos\frac{n\pi}{\lambda}(s-s_i)\sin\frac{n\pi}{\lambda}(s'-s_i) \\ &\times\sin\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\exp\frac{i}{2}(u-u'), \end{aligned} \tag{59}$$

and

$$\begin{aligned} G_{21} = &\ \theta(u-u')\frac{4i}{\lambda}\sum_{n=1}^{\infty}\frac{n\pi}{\lambda}\sin\frac{n\pi}{\lambda}(s-s_i)\cos\frac{n\pi}{\lambda}(s'-s_i) \\ &\times\sin\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\exp\frac{i}{2}(u-u'), \end{aligned} \tag{60}$$

and finally

$$\begin{aligned} G_{22} = &\ \theta(u-u')\frac{2i}{\lambda}\sum_{n=1}^{\infty}\sin\frac{n\pi}{\lambda}(s-s_i)\sin\frac{n\pi}{\lambda}(s'-s_i) \\ &\left\{\frac{1}{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}\sin\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\right. \\ &\left.-i\cos\left[\frac{\sqrt{\left(\frac{2n\pi}{\lambda}\right)^2+1}}{2}(u-u')\right]\right\}\exp\frac{i}{2}(u-u'). \end{aligned} \tag{61}$$

We can easily get the solution of the system (with field)

$$\begin{aligned} \overrightarrow{X}(s,u) = &\ \overrightarrow{X}_Q^{(0)}(s,u) \\ &+\int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u' G(s,u \mid s',u')\,\overrightarrow{w}(s',u'), \end{aligned} \tag{62}$$

and the solution $\overrightarrow{X}(s,u)$ can be written with the help of two basic vectors $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$:

$$\overrightarrow{X}(s,u) = \begin{pmatrix} x_Q \\ p_Q \end{pmatrix} = \overrightarrow{e_1}x_Q(s,u) + \overrightarrow{e_2}p_Q(s,u), \tag{63}$$

the expressions of x_Q and p_Q are obtained by using

$$\begin{aligned} x_Q(s,u) &= \overrightarrow{e_1}^{+}\overrightarrow{X}(s,u), \\ p_Q(s,u) &= \overrightarrow{e_2}^{+}\overrightarrow{X}(s,u). \end{aligned} \tag{64}$$

We notice that the field $A(\xi)$ is a function of the deviation $x_Q(s,u)$ and consequently the noise. Therefore, the field has a series expansion in the neighborhood of the classical path x_{cl}:

$$A|_{\eta(x_{\mathrm{cl}}+x_Q)} = \sum\nolimits_n \frac{1}{n!}(\eta x_Q)^n \left.\frac{\mathrm{d}^{n+1}A}{\mathrm{d}\xi^{n+1}}\right|_{\xi=\eta x_{\mathrm{cl}}}, \tag{65}$$

and by using (65), (50), (64) and the properties $\eta^2 = 0$, $\eta A = 0$, we obtain the following averages related to $\xi_Q = \eta x_Q$:

$$\begin{aligned} &\langle A(\xi)\rangle = A(\xi_{\mathrm{cl}}), \quad \langle A^2(\xi)\rangle = A^2(\xi_{\mathrm{cl}}), \\ &\langle p_{\mu Q}A^{\mu}(\xi)\rangle = 0, \quad \langle p_Q\rangle = 0, \end{aligned} \tag{66}$$

and

$$\begin{aligned} \langle p_Q^2\rangle = &\ 2\int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u' \\ &\times\left[G_{21}^2(s,u \mid s',u') + G_{22}^2(s,u \mid s',u')\right], \end{aligned} \tag{67}$$

and finally, using (35) we obtain

$$\langle H_Q\rangle = -\int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u'$$

$$\times \left(G_{21}^{2}\left(s,u\mid s',u'\right)+G_{22}^{2}\left(s,u\mid s',u'\right)\right); \tag{68}$$

therefore, we obtain the same result for the Hamiltonian average in the case of a free particle

$$\langle H_Q\rangle|_{A\neq 0}=\langle H_Q\rangle|_{A=0}\,. \tag{69}$$

Consequently, the fluctuation factor depends only on the Green's function $G\left(s,u\mid s',u'\right)$ related to the free Langevin system and the transition amplitude becomes

$$\begin{aligned}&\langle x_f,s_f\mid x_i,s_i\rangle\\&=c\exp\left[\mathrm{i}S_{\mathrm{cl}}\right]\exp\left[-\mathrm{i}\int^{s_i}\lim_{u\longrightarrow\infty}\int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u'\right.\\&\left.\times\left(G_{21}^{2}\left(s,u\mid s',u'\right)+G_{22}^{2}\left(s,u\mid s',u'\right)\right)\mathrm{d}s_i\right].\end{aligned} \tag{70}$$

Now, we fix the constant c by using (23) as $\lambda=(s_f-s_i)\to 0$

$$c=\frac{\mathrm{i}}{(2\pi)^2}. \tag{71}$$

3.3 Calculation in the configuration space

Preferably, we rely on the configuration space formalism for technical reasons. First, we give the Lagrangian for a spinless particle in interaction with a plane wave field

$$L\,\left[x\left(s\right),\dot{x}\left(s\right)\right]=-\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2-eA\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right), \tag{72}$$

then we separate the classical part L_{cl} from the other quantum one, L_Q, which collects all the remaining terms. It is easy to show that the Lagrangian has the following form:

$$\begin{aligned}\langle L_Q(s_i,u)\rangle&=-\frac{1}{2}\left\langle\frac{\partial x_Q(s_i,u)}{\partial s_i}\frac{\partial x_Q(s_i,u)}{\partial s_i}\right\rangle\\&-e\left\langle A\left(\xi\right)|_{\xi=\eta(x_{\mathrm{cl}}(s_i)+x_Q(s_i,u))}\frac{\partial x_Q(s_i,u)}{\partial s_i}\right\rangle,\end{aligned} \tag{73}$$

and that according to [12], the expression (73) becomes

$$\begin{aligned}&\langle L_Q\left(s_i\right)\rangle\\&=-\frac{1}{2}\lim_{s_1,s_2\to s_i}\partial_{s_1}\partial_{s_2}\langle x_Q\left(s_1,u\right)x_Q\left(s_2,u\right)\rangle\\&\quad-e\lim_{s_1,s_2\longrightarrow s_i}\left\langle A\left(\eta x_{\mathrm{cl}}\left(s_1\right)\right)\frac{\partial x_Q(s_2,u)}{\partial s_2}\right\rangle.\end{aligned} \tag{74}$$

The Hamiltonian average can be expressed as

$$\begin{aligned}\langle H_Q\left(s_i,u\right)\rangle=&-\left\langle p_Q\left(s_i,u\right)\frac{\partial x_Q(s_i,u)}{\partial s_i}\right\rangle\\&+\langle L_Q\left(s_i,u\right)\rangle,\end{aligned} \tag{75}$$

in addition to the fact that the stochastic quantum variables (x_Q,p_Q) are always governed by the Langevin system given in (49) and satisfy the following boundary conditions:

$$x_Q(s_i,u)=x_Q(s_f,u)=0. \tag{76}$$

At this level, we can show that the second term of the expression (74) vanishes when we use the field series expansion given in expression (65), the averages (66) and also the properties of the gauge and the white noise. Then we get the Lagrangian average

$$\langle L_Q\left(s_i,u\right)\rangle=-\frac{1}{2}\lim_{s_1,s_2\to s_i}\partial_{s_1}\partial_{s_2}\langle x_Q\left(s_1,u\right)x_Q\left(s_2,u\right)\rangle. \tag{77}$$

Now, the transition to a configuration space is carried out by using the Langevin equation of the momentum

$$\begin{aligned}\frac{\partial p_Q\left(s_i,u\right)}{\partial u}=&\,\mathrm{i}\left(\frac{\partial x_Q\left(s_i,u\right)}{\partial s_i}+\frac{\partial H_Q}{\partial p_Q\left(s_i,u\right)}\right)\\&+\chi(s_i,u),\end{aligned} \tag{78}$$

in addition to the results in (35), (50), (66) and the free Green's matrix as we take the limit $u\to\infty$, we finally obtain

$$\begin{aligned}&\lim_{u\to\infty}\left[\left\langle\frac{\partial x_Q\left(s_i',u\right)}{\partial s_i}p_Q\left(s_i,u\right)\right\rangle\right.\\&\left.+\left\langle\frac{\partial x_Q\left(s_i',u\right)}{\partial s_i}\frac{\partial x_Q\left(s_i,u\right)}{\partial s_i}\right\rangle\right]=0,\end{aligned} \tag{79}$$

(79) enables us to express the Hamiltonian average in the coordinate space according to the two-point correlation function $\langle x_Q\left(s_1,u\right)x_Q\left(s_2,u\right)\rangle$:

$$\begin{aligned}&\lim_{u\to\infty}\langle H_Q\left(s_i\right)\rangle\\&=\frac{1}{2}\lim_{s_1,s_2\to s_i}\partial_{s_1}\partial_{s_2}\lim_{u\to\infty}\langle x_Q\left(s_1,u\right)x_Q\left(s_2,u\right)\rangle.\end{aligned} \tag{80}$$

Then, the expression of the propagator relating to the plane wave is the same one as given by [4]

$$\begin{aligned}&\langle x_f,s_f\mid x_i,s_i\rangle\\&=c\exp\left[\mathrm{i}S_{\mathrm{cl}}\right]\,\exp\left[\frac{\mathrm{i}}{2}\int^{s_i}\lim_{s_1,s_2\to_s s_i}\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_1}\right.\\&\left.\qquad\times\lim_{u\to\infty}\langle x_Q(s_1,u)x_Q(s_2,u)\rangle\,\mathrm{d}s_i\right],\end{aligned} \tag{81}$$

where c is the constant given by (71). Thus, it remains to calculate the classical action S_{cl} and the two-point correlation function $\langle x_Q\left(s_1,u\right)x_Q\left(s_2,u\right)\rangle$.

3.4 Classical action calculation

Let us calculate the action according to the classical path

$$S=-\int_{s_i}^{s_f}\mathrm{d}s\left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2+eA\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\right], \tag{82}$$

we obtain by using the Euler–Lagrange equation the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{x}_\rho+eA_\rho)=e\dot{x}^\mu\partial_\rho A_\mu, \tag{83}$$

which gives us the equation of motion

$$\ddot{x} = eF\dot{x}. \tag{84}$$

We perform the first integration and obtain, using (83), the velocity given by

$$\dot{x}(s) = \dot{x}(s_i) + e\int_{s_i}^{s} \mathrm{d}s' f \frac{\mathrm{d}A}{\mathrm{d}\xi'}\dot{x}(s'); \tag{85}$$

then, we use an iterative treatment and obtain the solution in the same manner as in the phase space, with the help of the gauge properties $\eta f = 0$, $\eta^2 = 0$ and $\mathrm{d}A/\mathrm{d}\xi = F(\xi)$. Finally, we find the expression of the classical action

$$S^{\mathrm{cl}} = -\frac{(x_f - x_i)^2}{2\lambda} + \frac{e^2\lambda}{2(\xi_f - \xi_i)^2}\left[\int_{\xi_i}^{\xi_f} \mathrm{d}\xi A(\xi)\right]^2 - \frac{e^2\lambda}{2(\xi_f - \xi_i)}\int_{\xi_i}^{\xi_f} \mathrm{d}\xi\,[A(\xi)]^2. \tag{86}$$

3.5 Calculation of $\langle H_Q(s_i, u)\rangle$

First we separate the classical path x_{cl} from x ($x = x_{\mathrm{cl}} + y$), and the quantum action ($S_Q = S - S_{\mathrm{cl}}$) is now expressed as

$$S_Q = -\int_{s_i}^{s_f} \mathrm{d}s\left[\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}x_{\mathrm{cl}}}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) + eA(\xi)\left(\frac{\mathrm{d}x_{\mathrm{cl}}}{\mathrm{d}s} + \frac{\mathrm{d}y}{\mathrm{d}s}\right) - eA(\xi_{\mathrm{cl}})\cdot\frac{\mathrm{d}x_{\mathrm{cl}}}{\mathrm{d}s}\right]. \tag{87}$$

Next, we add the fictitious time u to the stochastic variable y, and the Langevin equation which governs $y(s,u)$ takes the following form:

$$\frac{\partial y(s,u)}{\partial u} = \mathrm{i}\frac{\partial^2}{\partial s^2}(y + x_{\mathrm{cl}}) - \mathrm{i}e\eta\left.\frac{\mathrm{d}A(\xi)}{\mathrm{d}\zeta}\right|_{\xi=\eta(x_{\mathrm{cl}}+x_Q)}(\dot{x}_{\mathrm{cl}} + \dot{y}) + \mathrm{i}e\frac{\partial A(\xi)}{\partial s} + \varrho(s,u), \tag{88}$$

with the following properties for the white noise:

$$\begin{cases} \langle\varrho^\mu(s,u)\rangle = 0, \\ \langle\varrho^\mu(s,u)\varrho^\nu(s',u')\rangle = 2g^{\mu\nu}\delta(s-s')\delta(u-u'). \end{cases} \tag{89}$$

Let us determine the solution of the Langevin equation. We can express it by a free one (without field) $y^{(0)}(s,u)$ and a remaining term $y'(s,u)$

$$y(s,u) = y^{(0)}(s,u) + y'(s,u), \tag{90}$$

and the free solution can be decomposed as

$$y^{(0)}(s,u) = y^{(0)}_{\mathrm{cl}}(s) + y^{(0)}_Q(s,u), \tag{91}$$

where $y^{(0)}_{\mathrm{cl}}(s)$ is the solution of the classical equation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}y^{(0)}_{\mathrm{cl}}(s) = 0, \tag{92}$$

directly, by using the boundary conditions $y^{(0)}_{\mathrm{cl}}(s_i) = y^{(0)}_{\mathrm{cl}}(s_f) = 0$, and the free classical solution is given by

$$y^{(0)}_{\mathrm{cl}}(s) = 0. \tag{93}$$

In addition, the free deviation $y^{(0)}_Q(s,u)$ is the solution of the following Langevin equation:

$$\frac{\partial}{\partial u}y^{(0)}_Q(s,u) = \mathrm{i}\frac{\partial^2}{\partial s^2}y^{(0)}_Q(s,u) + \varrho(s,u), \tag{94}$$

with the boundary conditions $y^{(0)}_Q(s_i,u) = y^{(0)}_Q(s_f,u) = 0$.

We can also note by $G^{(0)}(s,s';u-u')$ the free Green's solution of the following equation:

$$\left(\frac{\partial}{\partial u} + \frac{\partial^2}{\partial s^2}\right)G^{(0)}(s,s';u-u') = \delta(s-s')\delta(u-u'), \tag{95}$$

with the boundary conditions

$$G^{(0)}(s,s';u-u') = 0, \quad \text{for } s \text{ or } s' = s_i, s_f \text{ or } u < u'; \tag{96}$$

we calculate the free Green's function by using (95) and (96) and get the final expression

$$G^{(0)}(s,s';u-u') = \theta(u-u')\frac{2}{\lambda}\sum_{n=1}^{\infty}\sin\frac{n\pi}{\lambda}(s-s_i)\sin\frac{n\pi}{\lambda}(s'-s_i) \times \exp\left[\frac{\mathrm{i}n^2\pi^2}{\lambda^2}(u-u')\right]. \tag{97}$$

Now we use (94) and the expression (95) to get the free solution

$$y^{(0)}_Q(s,u) = \int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u' G^{(0)}(s,s';u-u')\varrho(s',u'); \tag{98}$$

therefore, the solution of the Langevin (88) in the presence of the plane wave $A_\mu(x)$ will take the following form:

$$y(s,u) = y^{(0)}_Q(s,u) + \mathrm{i}\int_{s_i}^{s_f}\mathrm{d}s'\int_{-\infty}^{+\infty}\mathrm{d}u' G^{(0)}(s,s';u-u') \times\left[\ddot{x}_{\mathrm{cl}}(s') + e\frac{\partial A}{\partial s'} - e\eta\frac{\mathrm{d}A}{\mathrm{d}\zeta'}(\dot{x}_{\mathrm{cl}}(s') + \dot{y}(s',u'))\right]. \tag{99}$$

We can notice that $y(s,u)$ still depends on $y(s',u')$; hence, we perform the calculations by an iterative treatment. Thanks to the properties $\eta^2 = 0$, $\eta A = 0$ and the fact that

$$\eta\ddot{x}_{\mathrm{cl}} = 0, \qquad \eta\dot{y}(s,u) = \eta\dot{y}^0_Q(s,u), \tag{100}$$

the solution $y(s,u)$ becomes

$$\begin{aligned} y(s,u) &= y_Q^{(0)}(s,u) \\ &+ \mathrm{i}\int_{s_i}^{s_f} \mathrm{d}s' \int_{-\infty}^{+\infty} \mathrm{d}u' G^{(0)}(s,s';u-u') \\ &\times \left[\ddot{x}_{\mathrm{cl}}(s') + e \left.\frac{\mathrm{d}A}{\mathrm{d}\zeta'}\right|_{\eta(x+y)} \eta\left(\dot{x}_{\mathrm{cl}}(s') + \dot{y}_Q^{(0)}(s',u')\right)\right] \\ &- \mathrm{i}e\eta \int_{s_i}^{s_f} \mathrm{d}s' \int_{-\infty}^{+\infty} \mathrm{d}u' G^{(0)}(s,s';u-u') \left.\frac{\mathrm{d}A}{\mathrm{d}\xi'}\right|_{\eta(x+y)} \\ &\times \left[\dot{x}_{\mathrm{cl}}(s') + \dot{y}_Q^{(0)}(s',u') \right. \\ &+ \mathrm{i}\int_{s_i}^{s_f} \mathrm{d}s'' \int_{-\infty}^{+\infty} \mathrm{d}u'' \frac{\partial}{\partial s'} G^{(0)}(s',s'';u'-u'') \\ &\times \left.\left(\ddot{x}_{\mathrm{cl}}(s'') + e\frac{\partial A}{\partial s''}\right)\right]. \end{aligned} \tag{101}$$

The two-point correlation function $\langle y(s_1,u_1).y(s_2,u_2)\rangle$ is also calculated by using the properties of the gauge $\eta^2 = 0$, $\eta A = 0, \eta\ddot{x} = 0$, (89) and (65). In addition we can show that

$$\eta y(s,u) = \eta y_Q^{(0)}(s,u), \tag{102}$$

and we deduce that the two-point correlation function in the presence of the field is the same one in the case of a free particle:

$$\langle y(s_1,u_1) y(s_2,u_2)\rangle = \left\langle y_Q^{(0)}(s_1,u_1) y_Q^{(0)}(s_2,u_2)\right\rangle. \tag{103}$$

First, we use (97) and perform the integration, and we obtain

$$\begin{aligned} &\left\langle y_Q^{(0)}(s_1,u_1) y_Q^{(0)}(s_2,u_2)\right\rangle \\ &= \sum_n \frac{8\mathrm{i}\lambda}{n^2\pi^2} \sin\frac{n\pi}{\lambda}(s_1-s_i) \\ &\times \left(\sin\frac{n\pi}{\lambda}(s_2-s_i) \exp\left[\frac{\mathrm{i}n^2\pi^2}{\lambda^2}|u_1-u_2|\right]\right); \end{aligned} \tag{104}$$

next, we take the limit $(u_1 = u_2 \to \infty)$ and use the formula [13]

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}, \qquad [0 \le x \le 2\pi]; \tag{105}$$

hence, the free two-point correlation function becomes

$$\begin{aligned} &\lim_{u_1=u_2\to\infty} \left\langle y_Q^{(0)}(s_1,u_1) y_Q^{(0)}(s_2,u_2)\right\rangle \\ &= \frac{4\mathrm{i}}{\lambda}\left[(s_2-s_i)\lambda - \left(s_1 s_2 - s_i(s_1+s_2) + s_i^2\right)\right]. \end{aligned} \tag{106}$$

Now, we impose $(\lambda = s_f - s_i)$ and obtain the two-point correlation function at the equilibrium $(u \to \infty)$

$$\lim_{u\to\infty} \langle y(s_1,u_1) y(s_2,u_2)\rangle = \frac{4\mathrm{i}}{\lambda}(s_2-s_i)(s_f-s_1), \tag{107}$$

and derive over the time s_1 and s_2 in (80), in order to obtain the Hamiltonian average as we take the limit $(u \to \infty)$. Finally, the calculation is simple and gives the following result:

$$\begin{aligned} &\lim_{u\to\infty} \langle H_Q(s_i,u)\rangle \\ &= \frac{1}{2} \lim_{s_1,s_2\to s_i} \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2} \lim_{u\to\infty} \langle y(s_1,u_1) y(s_2,u_2)\rangle \\ &= -\frac{2\mathrm{i}}{\lambda}. \end{aligned} \tag{108}$$

4 Green's function calculation

Owing to the configuration space, the calculations become easier, and we insert the results (86) and (71), and integrate over s using (26) and (108); thus, we obtain the propagator expression

$$\begin{aligned} \overline{K}(x_f,x_i;\lambda) = \frac{c}{\lambda^2} \exp\Bigg\{ &-\frac{\mathrm{i}m^2\lambda}{2} - \frac{\mathrm{i}(x_f-x_i)^2}{2\lambda} \\ &+ \frac{\mathrm{i}e^2\lambda}{2(\xi_f-\xi_i)^2}\left[\int_{\xi_i}^{\xi_f} \mathrm{d}\xi A(\xi)\right]^2 \\ &- \frac{\mathrm{i}e^2\lambda}{2(\xi_f-\xi_i)} \int_{\xi_i}^{\xi_f} \mathrm{d}\xi [A(\xi)]^2 \Bigg\}; \end{aligned} \tag{109}$$

finally, we insert these results in the expression (10). Therefore, we obtain the exact result for the Green's function in the case of a spinless particle in interaction with a plane wave field

$$\begin{aligned} \triangle(x_f,x_i) = \frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^2} \exp\Bigg\{ &-\frac{\mathrm{i}m^2\lambda}{2} - \frac{\mathrm{i}(x_f-x_i)^2}{2\lambda} \\ &+ \frac{\mathrm{i}e^2\lambda}{2(\xi_f-\xi_i)^2}\left[\int_{\xi_i}^{\xi_f} \mathrm{d}\xi A(\xi)\right]^2 \\ &- \frac{\mathrm{i}e^2\lambda}{2(\xi_f-\xi_i)} \int_{\xi_i}^{\xi_f} \mathrm{d}\xi [A(\xi)]^2 \Bigg\}. \end{aligned} \tag{110}$$

This result is equivalent to that given in [9] through the path integral approach.

5 Conclusion

In this paper, we have been able to calculate, within the framework of the stochastic quantization method, the exact and analytic Green function for a spinless particle in interaction with an electromagnetic plane wave field expressed in coordinate gauge using a perturbative treatment. We have solved iteratively the Langevin equation and obtained its solution under given boundary conditions. We notice that we have used in our work both phase and configuration spaces and applied a perturbative treatment. The calculations have been simplified using separately the classical and quantum contributions.

Consequently, this technique has constituted the most interesting stage in SQM, where we have collected all the quantum fluctuations in the same exponential propagator factor. Therefore, the exact expression of the Green's function has been determined when we take the equilibrium limit. Finally, this result agrees with that obtained via the path integral approach.

References

1. G. Parisi, Y.-S. Wu, Sci. Sin. **24**, 483 (1981)
2. P.H. Damgaard, H. Hüffel Stochastic quantization (World Scientific Publishing, 1988)
3. M. Namiki, Stochastic quantization (Springer, 1992)
4. H. Hüffel, P.H. Nakazato, Mod. Phys. Lett. A **9**, 2953 (1994)
5. K. Yuosa, H. Nakazato, hep-th/9610209 v1
6. A.N. Vaidya, C. Farina De Souza, Nuovo Cimento B **101**, 69 (1988)
7. J. Schwinger, Phys. Rev. **82**, 664 (1951)
8. A. Vaidya, C. Farina, M. Hott, Nuovo Cimento A **105**, 925 (1992)
9. T. Boudjedaa, L. Chetouani, L. Guechi, T.F. Hammann, Nuovo Cimento. B **109**, 219 (1994)
10. C. Cronsrtom, Phys. Lett. B **90**, 267 (1980)
11. L.S. Schulmann, Techniques and applications of path integration (Wiley, New York 1981)
12. R. Feynman, A. Hibbs, Quantum mechanics and path integrals (McGraw Hill, New York 1965)
13. I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series, and products (Academic Press, New York 1979)